

# Cooperative Games

## Lecture 1: Introduction

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## Why study coalitional games?

Cooperative games are a branch of game theory that models **cooperation** or collaboration between agents.

Coalitional games can also be studied from a computational point of view (e.g., the problem of succinct representation).

A coalition may represent a set of:

- persons or group of persons (labor unions, towns)
- objectives of an economic project
- artificial agents

We have a population  $N$  of  $n$  agents.

**Definition** (Coalition)

A **coalition**  $\mathcal{C}$  is a set of agents:  $\mathcal{C} \subseteq 2^N$ .

## The classic problem

- $N$  is the set of all agents (or players)
- $v: 2^N \rightarrow \mathbb{R}$  is the **valuation function**. For  $\mathcal{C} \subseteq N$ ,  $v(\mathcal{C})$  is the value obtained by the coalition  $\mathcal{C}$

**problem:** a game  $(N, v)$ , and we assume all agents in  $N$  want to cooperate.

**solution:** a **payoff distribution**  $x \in \mathbb{R}^n$  that provides a value to individual agents.

What are the interesting **properties** that  $x$  should satisfy?

How to **determine** the payoff vector  $x$ ?

## An example

$$\begin{aligned} N &= \{1, 2, 3\} \\ v(\{1\}) &= 0, v(\{2\}) = 0, v(\{3\}) = 0 \\ v(\{1, 2\}) &= 90 \\ v(\{1, 3\}) &= 80 \\ v(\{2, 3\}) &= 70 \\ v(\{1, 2, 3\}) &= 105 \end{aligned}$$

What should we do?

- form  $\{1, 2, 3\}$  and share equally  $\langle 35, 35, 35 \rangle$ ?
- 3 can say to 1 "let's form  $\{1, 3\}$  and share  $\langle 40, 0, 40 \rangle$ ".
- 2 can say to 1 "let's form  $\{1, 2\}$  and share  $\langle 45, 45, 0 \rangle$ ".
- 3 can say to 2 "OK, let's form  $\{2, 3\}$  and share  $\langle 0, 46, 24 \rangle$ ".
- 1 can say to 2 and 3, "fine!  $\{1, 2, 3\}$  and  $\langle 33, 47, 25 \rangle$ "
- ... is there a "good" solution?

## 1- Games with Transferable Utility (TU games)

- Any two agents can **compare** their utility
- Utility can be **transferred** between agents

**Definition** (valuation or characteristic function)

A *valuation function*  $v$  associates a real number  $v(\mathcal{C})$  to any subset  $\mathcal{C} \subseteq N$ , i.e.,  $v: 2^N \rightarrow \mathbb{R}$

Without loss of generality, we will assume that  $\begin{cases} v(\emptyset) = 0 \\ v(\mathcal{C}) \geq 0 \end{cases}$

**Definition** (TU game)

A TU game is a pair  $(N, v)$  where  $N$  is a set of agents and where  $v$  is a valuation function.

## 2- Games with Non Transferable Utility (NTU games)

It is **not** always possible to compare the utility of two agents or to transfer utility (e.g., no price tags). Agents have preference over coalitions.

## Today

- We provide some examples of TU games.
- We discuss some desirable solution properties.
- We end with a quick overview of the course and practicalities

## Informal example: a task allocation problem

- A set of tasks needs to be performed,
  - they require different expertises
  - they may be decomposed.
- Agents do not have enough resource on their own to perform a task.
- Find complementary agents to perform the tasks
  - robots have the ability to move objects in a plant, but multiple robots are required to move a heavy box.
  - transportation domain: agents are trucks, trains, airplanes, ships... a task is a good to be transported.
- Issues:**
  - What coalition to form?
  - How to reward each member when a task is completed?

## Market games (classic application in Economics)

A **market** is a quadruple  $(N, M, A, F)$  where

- $N$  is a set of traders
- $M$  is a set of  $m$  continuous good
- $A = (a_i)_{i \in N}$ ,  $a_i \in \mathbb{R}_+^m$  is the initial endowment vector
- $F = (f_i)_{i \in N}$ ,  $f_i: \mathbb{R}_+^m \rightarrow \mathbb{R}$  is the valuation function vector

Assumptions of the model:

- The **utility** of agent  $i$  for possessing  $x \in \mathbb{R}_+^m$  and an amount of money  $p \in \mathbb{R}$  is  $u_i(x, p) = f_i(x) + p$ . The money models side payments.
- Initially, agents have **no money**.
- $p_i$  can be **positive** or **negative** (like a bank account).
- Agents can increase their utility by **trading**: after a trade among the members of  $S$ , they have an endowment  $(b_i)_{i \in S}$  and money  $(p_i)_{i \in S}$  such that  $\sum_{i \in S} a_i = \sum_{i \in S} b_i$  and  $\sum_{i \in S} p_i = 0$ .

## Market games (cont.)

### Definition (Market game)

A game  $(N, v)$  is a market game if there exists a market  $(N, M, A, F)$  such that, for every  $S \subseteq N$ ,

$$v(S) = \max \left\{ \sum_{i \in S} f_i(x_i) \mid x_i \in \mathbb{R}_+^m, \sum_{i \in S} x_i = \sum_{i \in S} a_i \right\}$$

Shapley. The solutions of a symmetric market game, in Contributions to the Theory of Games, Luce and Tuckers editors, 1959

Shapley and Shubik. On market games, *Journal of Economic Theory*, 1, 9-25, 1969

## Cost allocation games

### Definition (Cost allocation game)

A cost allocation game is a game  $(N, c)$  where

- $N$  represents the potential customers of a public service or a public facility.
- $c(S)$  is the cost of serving the members of  $S$

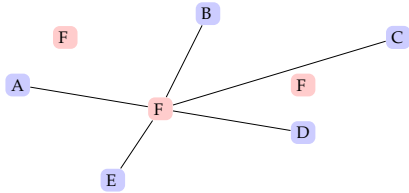
Mathematically speaking, a cost game is a game. The special status comes because of the different intuition (worth of a coalition vs. cost of a coalition).

We can associate a cost game with a "traditional game" using the corresponding saving game  $(N, v)$  given by

$$v(S) = \sum_{i \in S} c(\{i\}) - c(S).$$

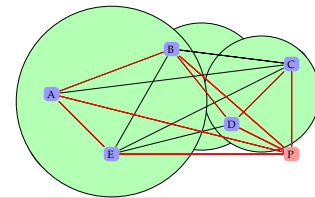
## Examples of cost allocation games

- **Airport game:**  $n$  types of planes can land on a runway. The cost to accommodate a plane of type  $k$  is  $c_k$ . The cost is defined as  $c(S) = \max_{k \in S} \{c_k\}$
- **Sharing a water supply system:**  $n$  towns considers building a common water treatment facility. The cost of a coalition is the minimum cost of supplying the coalition members by the most efficient means.



## Examples of cost allocation games

- **Minimum cost spanning tree games:** a set  $H$  of houses have to be connected to a power plant  $P$ . The houses can be linked directly to  $P$  or to another house. The cost of connecting two locations  $(i, j) \in H \cup \{P\}$  is  $c_{ij}$ . Let  $S \subseteq H$ .  $\Gamma(S)$  is the minimum cost spanning tree spanning over the set of edges  $S \cup \{P\}$ . The cost function is  $c(S) = \sum_{\text{all edges of } \Gamma(S)} c_{ij}$ .



## Simple or Voting games

### Definition (voting games)

A game  $(N, v)$  is a **voting game** when

- the valuation function takes two values
- 1 for a winning coalitions
- 0 for the losing coalitions

$v$  satisfies *unanimity*:  $v(N) = 1$

$v$  satisfies *monotonicity*:  $S \subset T \Rightarrow v(S) \leq v(T)$

## Weighted Voting games

### Definition (weighted voting games)

A game  $(N, w_{i \in N}, q, v)$  is a **weighted voting game** when  $v$  satisfies unanimity, monotonicity and the valuation function is defined as

$$v(S) = \begin{cases} 1 & \text{when } \sum_{i \in S} w_i \geq q \\ 0 & \text{otherwise} \end{cases}$$

**Example:** 1958 European Economic Community: Belgium, Italy, France, Germany, Luxembourg and the Netherlands. Each country gets the following number of votes:

- Italy, France, Germany: 4
- Belgium, the Netherlands: 2
- Luxembourg: 1

The threshold of the game is  $q = 12$ .

## Some types of TU games

$$\forall \mathcal{C}_1, \mathcal{C}_2 \subseteq N \mid \mathcal{C}_1 \cap \mathcal{C}_2 = \emptyset, i \in N, i \notin \mathcal{C}_1$$

- **additive (or inessential):**  $v(\mathcal{C}_1 \cup \mathcal{C}_2) = v(\mathcal{C}_1) + v(\mathcal{C}_2)$   
trivial from the game theoretic point of view
- **superadditive:**  $v(\mathcal{C}_1 \cup \mathcal{C}_2) \geq v(\mathcal{C}_1) + v(\mathcal{C}_2)$  satisfied in many applications: it is better to form larger coalitions.
- **weakly superadditive:**  $v(\mathcal{C}_1 \cup \{i\}) \geq v(\mathcal{C}_1) + v(\{i\})$
- **subadditive:**  $v(\mathcal{C}_1 \cup \mathcal{C}_2) \leq v(\mathcal{C}_1) + v(\mathcal{C}_2)$
- **convex:**  $\forall \mathcal{C} \subseteq \mathcal{T}$  and  $i \notin \mathcal{T}$ ,  
 $v(\mathcal{C} \cup \{i\}) - v(\mathcal{C}) \leq v(\mathcal{T} \cup \{i\}) - v(\mathcal{T})$ .  
Convex game appears in some applications in game theory and have nice properties.
- **monotonic:**  $\forall \mathcal{C} \subseteq \mathcal{T} \subseteq N$   $v(\mathcal{C}) \leq v(\mathcal{T})$ .

## Some properties

Let  $x \in \mathbb{R}^n$  be a solution of the TU game  $(N, v)$

**Feasible solution:**  $\sum_{i \in N} x(i) \leq v(N)$

**Anonymity:** a solution is independent of the names of the player.

### Definition (marginal contribution)

The **marginal contribution** of agent  $i$  for a coalition  $\mathcal{C} \subseteq N \setminus \{i\}$  is  $mc_i(\mathcal{C}) = v(\mathcal{C} \cup \{i\}) - v(\mathcal{C})$ .

Let  $mc_i^{\min}$  and  $mc_i^{\max}$  denote the minimal and maximal marginal contribution.

$x$  is **reasonable from above** if  $\forall i \in N$   $x^i \leq mc_i^{\max}$

$\Leftrightarrow mc_i^{\max}$  is the strongest **threat** that an agent can use against a coalition.

$x$  is **reasonable from below** if  $\forall i \in N$   $x^i \geq mc_i^{\min}$

$\Leftrightarrow mc_i^{\min}$  is a minimum acceptable reward.

### Some properties

Let  $x, y$  be two solutions of a TU-game  $(N, v)$ .

**Efficiency:**  $x(N) = v(N)$

⇔ the payoff distribution is an allocation of the entire worth of the grand coalition to all agents.

**Individual rationality:**  $\forall i \in N, x(i) \geq v(\{i\})$

⇔ agent obtains at least its self-value as payoff.

**Group rationality:**  $\forall \mathcal{C} \subseteq N, \sum_{i \in \mathcal{C}} x(i) = v(\mathcal{C})$

⇔ if  $\sum_{i \in \mathcal{C}} x(i) < v(\mathcal{C})$  some utility is lost

⇔ if  $\sum_{i \in \mathcal{C}} x(i) > v(\mathcal{C})$  is not possible

**Pareto Optimal:**  $\sum_{i \in N} x(i) = v(N)$

⇔ no agent can improve its payoff without lowering the payoff of another agent.

An **imputation** is a payoff distribution  $x$  that is efficient and individually rational.

### Summary

- Two main classes of games: TU games and NTU games
- Examples of TU games: market games, cost allocation games, voting games
- Some classes of TU games: superadditive, convex, etc.
- Some desirable properties of a solution

### Coming next

A first solution concept to ensure stable coalitions: the **core**.

**Definition** (Core for superadditive games)

The **core** of a game  $(N, v)$  is the set:

$$\{x \in \mathbb{R}^n \mid x(S) \geq v(S) \text{ for all } S \subseteq N\}$$

### Course overview

- Game theory stability concepts: the core, the nucleolus, the kernel
- A fair solution concept: the Shapley value
- Special types of games: Voting games
- Representation and complexity
- Other model of cooperation: NTU games and hedonic games.
- Issues raised by practical approaches (search for optimal CS, uncertainty, overlapping coalition, etc).

### Practicalities

- Webpage: <http://staff.science.uva.nl/~stephane/Teaching/CoopGames/2012/>  
It will contain the lecture notes and the slides, posted shortly before class.
- Evaluation:  $6\text{ECTS} = 6 \cdot 28\text{h} = 168\text{h}$ .
  - some homeworks (every two or three weeks) 40% of the grade.  $\LaTeX$  is preferred, but you can hand-write your solution.
  - final paper 50% of the grade (more details at the end of the first block)
  - final presentation 10% of the grade
  - no exam.**
- Attendance: not part of the grade.